Quantum field theory on manifolds with a boundary

2005 J. Phys. A: Math. Gen. 38 10393

(http://iopscience.iop.org/0305-4470/38/48/010)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 10393–10401 doi:10.1088/0305-4470/38/48/010

Quantum field theory on manifolds with a boundary

Z Haba

Institute of Theoretical Physics, University of Wroclaw, 50-204 Wroclaw, Plac Maxa Borna 9, Poland

E-mail: zhab@ift.uni.wroc.pl

Received 18 May 2005, in final form 15 September 2005
Published 16 November 2005
Online at stacks.iop.org/JPhysA/38/10393

Abstract

We discuss quantum theory of fields ϕ defined on a $(d + 1)$-dimensional manifold $\mathcal{M}$ with a boundary $\mathcal{B}$. The free action $W_0(\phi)$ which is a bilinear form in ϕ defines the Gaussian measure with a covariance (Green function) $\mathcal{G}$. We discuss a relation between the quantum field theory with a fixed boundary condition Φ and the theory defined by the Green function $\mathcal{G}$. It is shown that the latter results by an average over Φ of the first. The QFT in anti-de Sitter space is treated as an example. It is shown that quantum fields on the boundary are more regular than those on anti-de Sitter space.

PACS numbers: 04.62+v, 02.50Cw

1. Introduction

Quantum field theory (QFT) can be defined by a functional integral

$$d\mu(\phi) = \mathcal{D}\phi \exp(-W(\phi)) \tag{1}$$

where W is the classical action. From the point of view of formal properties (translational invariance) of such a functional integral it should not matter whether we write in it ϕ or $\phi + \phi_0$. However, if the action is defined on a manifold with a boundary then the dependence on the boundary value of ϕ seems to be crucial [1–3]. This means that a formal invariance under translations in function space $\phi \to \phi + \phi_0$ must be broken in the definition of the functional integral in [1–3]. Then, the dependence on the boundary value breaks some symmetries present in the classical action W. Such an approach to QFT disagrees with the conventional one based on the mode summation [4–11] or perturbation expansion in the number N of components or in the coupling constant. In this paper, we discuss a relation between the two approaches in the framework of the functional integral. In the anti-de Sitter models of [1–3] the boundary appears at the spatial infinity and coincides with the (compactified) Minkowski space. In the Euclidean version of the anti-de Sitter space (in the Poincare coordinates) the boundary can be realized as the Euclidean subspace of the hyperbolic space.

We consider a Riemannian manifold $\mathcal{M}$ with the boundary $\mathcal{B}$. The metric on $\mathcal{M}$ is denoted by G and its restriction to $\mathcal{B}$ by g. We shall denote the coordinates on $\mathcal{M}$ by X and their restriction to $\mathcal{B}$ by x; close to the boundary we write $X = (y, x)$. The action for a minimally coupled massless free scalar field ϕ reads

$$W_0(\phi) = \int_{\mathcal{M}} \mathrm{d}X\, \sqrt{G} G^{AB} \partial_A\phi\partial_B\phi \equiv (\phi, \mathcal{A}\phi). \tag{2}$$

The non-negative bilinear form (2) is defined on a certain domain $D(\mathcal{A})$ of functions. Such a bilinear form determines a self-adjoint operator $\mathcal{A}$ [12] (the definition of $\mathcal{A}$ depends on the choice of $D(\mathcal{A})$) in the Hilbert space of square integrable functions. The free (Euclidean) quantum field ϕ is defined by $\mathcal{A}^{-1}$ in the sense that the kernel of $\mathcal{A}^{-1}$ (the Green function) provides a definition of the two-point correlation function of ϕ. The Green function $\mathcal{G}$ is a solution of the equation

$$-\mathcal{A}\mathcal{G} \equiv \partial_A G^{AB}\sqrt{G}\partial_B\mathcal{G} = \delta \tag{3}$$

where $G = \det G_{AB}$. The solution of equation (3) is not unique. If $\mathcal{G}'$ is another solution of equation (3) then $\mathcal{G}' = \mathcal{G} + \mathcal{S}$ where $\mathcal{S}$ is a solution of the equation

$$\mathcal{A}\mathcal{S} = 0. \tag{4}$$

We can determine $\mathcal{G}$ unambiguously imposing some additional requirements, for example, requiring that $\mathcal{G} = 0$ on the boundary. The various definitions of $\mathcal{G}$ correspond to various choices of $D(\mathcal{A})$ in the definition of the bilinear form (2) [12, 13].

2. The functional measure

To the free action (2) we add a local interaction V. Now, the total action reads

$$W = W_0 + W_I = \int_{\mathcal{M}} \mathrm{d}X\, \sqrt{G} G^{AB}\partial_A\phi\partial_B\phi + \int_{\mathcal{M}} \mathrm{d}X\, \sqrt{G} V(\phi). \tag{5}$$

We can give a mathematical definition of the formal functional measure (1)

$$\mathrm{d}\mu_V(\phi) = Z_0^{-1}\, \mathrm{d}\mu_0(\phi) \exp(-W_I) \tag{6}$$

where the Gaussian measure μ_0 is a mathematical realization of the formal integral

$$\mathrm{d}\mu_0(\phi) = \mathcal{D}\phi \exp(-W_0).$$

The partition function Z_0 in equation (6)

$$Z_0 = \int \mathrm{d}\mu_0 \exp(-W_I) \tag{7}$$

determines a normalization factor.

We do not discuss in this paper some divergence problems which may arise if V is a local function of ϕ and $\mathcal{M}$ has an infinite volume. We may assume that W_I has been properly regularized. We have a suggestion how to construct a regular QFT at the end of this paper.

A functional measure μ defines a probability distribution of fields $\phi(X)$; more precisely the 'smeared out' fields

$$(\phi, f) = \int \mathrm{d}X\, \sqrt{G}\phi(X) f(X).$$

In probability theory (see, e.g., [14, 15]) it is convenient to treat the probability measure μ defined on some sets of random fields ϕ as one of many possible realizations of the probability space (Ω, Σ, P). The random field $\phi : \Omega \to R$ (at fixed X) is a map from the set Ω to the set of real numbers such that the two-point correlation function is an average over the 'sample paths' $\omega \in \Omega$,

$$\langle \phi(X)\phi(Y)\rangle = \int_\Omega \mathrm{d}P(\omega)\phi_\omega(X)\phi_\omega(Y)$$

where P is a probability measure on the σ-algebra Σ of subsets of Ω. The Gaussian measure gives a realization of the Gaussian random field ϕ_ω. It is defined [14, 16] by the mean

$$m(X) = \int \mathrm{d}\mu(\phi)\phi(X) \equiv \langle \phi(X)\rangle$$

and the covariance

$$\begin{aligned}\mathcal{G}(X, Y) &= \int \mathrm{d}\mu(\phi)(\phi(X) - \langle \phi(X)\rangle)(\phi(Y) - \langle \phi(Y)\rangle)\\ &= \langle(\phi(X) - \langle \phi(X)\rangle)(\phi(Y) - \langle \phi(Y)\rangle)\rangle \end{aligned} \tag{8}$$

or by its characteristic function S,

$$S[if] = \int \mathrm{d}\mu \exp(\mathrm{i}(\phi, f)) = \exp(\mathrm{i}(m, f) - \frac{1}{2}(f, \mathcal{G}f)).$$

Note that if we make a shift in the function space and define $\tilde{\phi} = \phi - m$ then $\tilde{\phi}$ has zero mean. Hence, we could subtract the mean value defining a new Gaussian measure

$$\mathrm{d}\tilde{\mu}(\tilde{\phi}) = \mathrm{d}\mu(\tilde{\phi} + m).$$

The Gaussian measure is quasi-invariant under a shift χ if there exists an integrable function $\rho(\phi, \chi)$ such that

$$\mathrm{d}\mu(\phi + \chi) = \mathrm{d}\mu(\phi)\rho(\phi, \chi). \tag{9}$$

It is easy to see by a calculation of the characteristic function of both sides of equation (9) that the measure μ is quasi-invariant under the shift χ if [16]

$$\rho(\phi, \chi) = \exp(-(\phi, B\chi) - \tfrac{1}{2}(\chi, C\chi)) \tag{10}$$

and the following equations are satisfied:

$$\chi = \mathcal{G}B\chi \tag{11}$$

$$(B\chi, \mathcal{G}B\chi) = (\chi, C\chi). \tag{12}$$

Equations (9)–(12) express the formal invariance of the functional measure (1) under translations in the function space. If these conditions are not satisfied then it really does matter what is the shift χ. In some papers on AdS-CFT correspondence [1–3, 11, 17] the choice is made $\mathcal{G}(X, Y) = \mathcal{G}_D(X, Y)$ where $\mathcal{G}_D$ is the Dirichlet Green function (vanishing on the boundary) and $\langle \phi(X)\rangle = \phi_0(X)$ where $\phi_0(X)$ is a solution of the equation

$$\mathcal{A}\phi_0 = 0 \tag{13}$$

with a fixed boundary condition Φ. We can see that equations (11) and (12) cannot be satisfied if $\chi = \phi_0$. Hence, the partition function $Z[\Phi]$ may depend on the boundary value Φ.

In general, choosing in QFT the boundary field $\phi_0 \neq 0$ we break some symmetries of the classical action (5). As an example we could consider the hyperbolic space with the metric

$$\mathrm{d}s^2 = y^{-2}\big(\mathrm{d}y^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_d^2\big). \tag{14}$$

The hyperbolic space (14) has compactified R^d as the boundary [3]. The hyperbolic space can be considered as a Euclidean version of anti-de Sitter space (AdS_{d+1} has compactified Minkowski space as a boundary at conformal infinity [3]). It is also a Euclidean version of de Sitter space. However, the Poincare coordinates (14) are inappropriate for an analytic continuation of quantum fields from the hyperbolic space to de Sitter space (there is also no boundary at conformal infinity of de Sitter space).

The action (5) in the hyperbolic space is invariant under R^d rotations and translations whereas the quantum field theory with a fixed boundary value of ϕ_0 would not be invariant under these symmetries. The approach to QFT assuming a boundary condition Φ for the field ϕ_0 and the Dirichlet boundary condition for the Green function leads to a different quantum field theory than that developed in [5–8]. The latter is determined by the mean $\langle\phi\rangle = 0$ and a choice of the Green function (the free propagator $\mathcal{G}$ solving equation (3)) which does not vanish on the boundary. A possible way to determine the propagator is to construct it for a real time by a mode summation and subsequently to continue analytically the propagator to the imaginary time (for a class of models this is done in [4, 6]; the mode summation is also not unique). It seems reasonable to choose the Green function $\mathcal{G}$ which has the symmetries of the action W_0 (1) as in [4–7, 11]. Then, the functional measure (6) will have the symmetries of the action (5).

3. An average over the boundary values

After the heuristic discussion in section 2 of functional integration over fields with a fixed boundary value we prove in this section that the approach starting from the free propagator $\mathcal{G}$ is equivalent to a quantization around a classical solution ϕ_0 with a prescribed boundary value Φ if subsequently an average over all such boundary values is performed. First, let us assume (in the sense that for the bilinear forms $(f, \mathcal{G}f) \geqslant (f, \mathcal{G}_D f)$)

$$\mathcal{G} \geqslant \mathcal{G}_D. \tag{15}$$

If the operator $\mathcal{A}$ is an elliptic operator then inequality (15) follows from the maximum principle for elliptic operators [18, 19]. We are also interested in operators $\mathcal{A}$ with singular or vanishing coefficients which need not be elliptic. It is not clear whether the inequality (15) can be satisfied for such operators. However, inequality (15) still holds true for the Green functions of singular operators discussed in [20, 21] which are expressed by a path integral. The Dirichlet condition imposes a restriction on the class of paths. Hence, the integral over a restricted set of paths is bounded by $\mathcal{G}$ in equation (15).

If inequality (15) is satisfied then there exists a positive definite bilinear form $\mathcal{G}_B$ such that

$$\mathcal{G}(X, X') = G_D(X, X') + \mathcal{G}_B(X, X'). \tag{16}$$

Clearly, on the boundary

$$\mathcal{G}(0, x; 0, x') = \mathcal{G}_B(0, x; 0, x') \equiv \mathcal{G}_E(x, x') \tag{17}$$

$\mathcal{G}_E$ defines a non-negative bilinear form on the set of functions defined on the boundary $\mathcal{B}$.

Theorem 1. *Let μ_0 be the Gaussian measure with the mean zero and the covariance $\mathcal{G}$. Assume that $\mathcal{G}$ and $\mathcal{G}_D$ are real positive definite bilinear forms satisfying inequality (15). Then, there exist independent Gaussian random fields ϕ_D and ϕ_B with the mean equal to zero and the covariance $\mathcal{G}_D$ and $\mathcal{G}_B$ respectively such that for any integrable function* $\exp(-W_I)F$

$$\int \mathrm{d}\mu_0(\phi) \exp(-W_I(\phi))F(\phi) = \int \mathrm{d}\mu_D(\phi_D)\, \mathrm{d}\mu_B(\phi_B) \exp(-W_I(\phi_D + \phi_B))F(\phi_D + \phi_B). \tag{18}$$

In this sense

$$\phi = \phi_D + \phi_B. \tag{19}$$

The theorem and its proof can be found in [13, 14]. It is easy to check equation (18) for the generating functional (then $\exp(-W_I(\phi))F(\phi) = \exp(\phi, J)$). On a perturbative level the general formula (18) follows from the one for the generating functional. For the general theory of 'conditioning' (15) see [13, 14]. Equation (18) is discussed in the lattice approximation in [14] (section 8.1). Another derivation and a discussion of its relevance to the AdS-CFT correspondence can be found in [22]. The relevance of an average over the boundary values for the Hamiltonian formulation of the quantum field theory is discussed in [23].

Let us note that on a formal level

$$\mathrm{d}\mu_D(\phi_D) = \mathcal{D}\phi_D \exp\left(-\tfrac{1}{2}(\phi_D, \mathcal{A}_D\phi_D)\right)$$

where $\mathcal{A}_D$ is the Laplace–Beltrami operator with the Dirichlet boundary conditions. On a formal level $\mathcal{A}_D\phi_0 = 0$. Hence, $\mathrm{d}\mu_D(\phi_D + \phi_0) = \mathrm{d}\mu_D(\phi_D)$ although strictly speaking the shift of μ_D by ϕ_0 does not make sense because ϕ_0 does not vanish on the boundary. We treat the rhs of equation (18) (before an integration over ϕ_B) as a rigorous version of the QFT shifted by a classical solution. This interpretation is suggested by

Theorem 2. *Let $\mathcal{G}_D$ be the Dirichlet Green function of the operator $\mathcal{A}$ (equation (3)). Let $\mathcal{G}$ be another real solution of equation (3) satisfying inequality (15). Then, there exists a Gaussian random field Φ defined on the boundary $\mathcal{B}$ with the mean zero and the covariance $\mathcal{G}_E$ such that (in the sense of $L^2(\mathrm{d}P)$ integrals [15])*

$$\phi_B(X) = \int_{\mathcal{B}} \mathrm{d}x_b \sqrt{g}\mathcal{D}(X, x_b)\Phi(x_b) \tag{20}$$

where $\mathcal{D}(X, x_b)$ is the Green function solving the boundary value problem for equation (13).

Proof. Let us note that $\mathcal{G}_D$ as well as $\mathcal{G}$ satisfy the same equation (3). Then, their difference $\mathcal{G}_B = \mathcal{G} - \mathcal{G}_D$ satisfies the equations

$$\mathcal{A}(X)\mathcal{G}_B(X, X') = \mathcal{A}(X')\mathcal{G}_B(X, X') = 0 \tag{21}$$

and the boundary condition $\mathcal{G}_B(0, x; 0, x') = \mathcal{G}_E(x, x')$. We can solve equation (21) with the given boundary condition $\mathcal{G}_E$

$$\mathcal{G}_B(X, X') = \int_{\mathcal{B}} \mathrm{d}x_b \sqrt{g}\mathcal{D}(X, x_b) \int_{\mathcal{B}} \mathrm{d}x'_b \sqrt{g}\mathcal{D}(X', x'_b)\mathcal{G}_E(x_b, x'_b) \tag{22}$$

where $\mathcal{D}$ is the Green function solving the Dirichlet boundary problem for equation (13). The bilinear form $\mathcal{G}_E$ (17) defines a Gaussian field Φ on the probability space (Ω, Σ, P) (see section 2) with the mean zero and the covariance

$$\langle\Phi(x)\Phi(x')\rangle = \mathcal{G}_E(x, x'). \tag{23}$$

We define $\tilde{\phi}_B$ by the rhs of equation (20) where the integral can be understood in the sense of the $L^2(\mathrm{d}P)$ convergence of the Riemann sums (see, e.g., [15]). For the proof of the theorem ($\tilde{\phi} = \phi$) it is sufficient to show that the covariance of $\tilde{\phi}_B$ coincides (as a bilinear form) with $\mathcal{G}_B$. This is a consequence of equation (22). □

Let us note that there exists the Gaussian measure ν_B such that

$$\int \mathrm{d}\nu_B(\Phi)\Phi(x)\Phi(x') = \mathcal{G}_E(x, x').$$

ν_B can be defined by μ_B as $\nu_B = \mu_B \circ T$ where $T(\Phi) = \phi_B$ is the one-to-one map (20) expressing the solution of equation (13) by its boundary value. We can see that if there is a QFT with a two-point function $\mathcal{G}$ non-vanishing on the boundary then there is the unique choice of ϕ_0 solving equation (13) such that $\phi = \phi_D + \phi_0$ is a realization of a random field with the boundary value Φ. An average over Φ leads to the Green function $\mathcal{G}$.

In the example of the hyperbolic space (14) (with the Poincare coordinates) the solution (20) of the Dirichlet boundary problem (13) can be expressed by its boundary value $\phi_B(y=0,x) = \Phi(x)$ [17]

$$\phi_B(X) = y^{\frac{d}{2}} \int \mathrm{d}p \exp(\mathrm{i}px)|p|^{\frac{d}{2}} K_{\frac{d}{2}}(|p|y)\tilde{\phi}(p) \tag{24}$$

where $\tilde{\phi}$ denotes the Fourier transform of Φ and K_ν is the modified Bessel function of order ν [29]. Comparing with equation (20) we obtain $(X = (y, x))$

$$\mathcal{D}(X, x') = (2\pi)^{-d} y^{\frac{3}{2}d+1} \int \mathrm{d}p \exp(\mathrm{i}p(x-x'))|p|^{\frac{d}{2}} K_{\frac{d}{2}}(|p|y).$$

The two-point function $\mathcal{G}_E$ resulting from the QFT on the hyperbolic space constructed in [4, 6, 10] is $\mathcal{G}_E = -\ln|x-x'|$ [9, 24–26]. In the Fourier transforms (up to an inessential normalization) we have (see the discussion in [4, 20, 21, 25])

$$\langle\tilde{\phi}(p)\tilde{\phi}^*(p')\rangle = \delta(p-p')|p|^{-d}. \tag{25}$$

We may apply equation (25) to calculate $\langle\phi_B(X)\phi_B(X')\rangle$. As a solution of equation (21) after an analytic continuation to the real time it must coincide with the Hadamard two-point function (vacuum expectation value of an anticommutator of quantum scalar fields) which is usually denoted by $G^{(1)}(X, X')$ (the formula for $\mathcal{G}_B$ in the hyperbolic space can be found in [26] and for $\mathcal{G}_D$ in [30])

$$\mathcal{G}_B(X, X') = \langle\phi_B(X)\phi_B(X')\rangle = (yy')^{\frac{d}{2}} \int \mathrm{d}p \exp(\mathrm{i}p(x-x'))K_{\frac{d}{2}}(|p|y)K_{\frac{d}{2}}(|p|y'). \tag{26}$$

4. Non-linear boundary value problem

We can modify the formulation (6)–(13) of QFT on manifolds with a boundary so that the interaction $V(\phi)$ is taken into account already at the classical level. Then, instead of equation (13) we consider the equation

$$-\mathcal{A}\psi = V'(\psi) \tag{27}$$

or in the integral form

$$\psi(X) = \phi_B(X) + \int \mathrm{d}X' \sqrt{G}\mathcal{G}_D(X, X')V'(\psi(X')) \tag{28}$$

where ϕ_B is defined in equation (20). In order to express the functional integral (6) in terms of ψ let us introduce a differential operator

$$\mathcal{A}_\psi = \mathcal{A} + V''(\psi). \tag{29}$$

Define $\mathcal{G}_D^\psi$ as the Dirichlet Green function of $\mathcal{A}_\psi$. Let μ_ψ be the Gaussian measure with the mean zero and the covariance $\mathcal{G}_D^\psi$. Then, the formula (18) reads (under the assumption that the function on the rhs of equation (30) is integrable)

$$\int \mathrm{d}\mu_0(\phi) \exp(-W_I(\phi))F(\phi) = \int \mathrm{d}\mu_B(\phi_B) \exp(W_0(\phi_B) - W(\psi)) \det(\mathcal{A}_\psi)^{-\frac{1}{2}} \det(\mathcal{A})^{\frac{1}{2}}$$
$$\int \mathrm{d}\mu_\psi(\phi_D) \exp\left(-\int \mathrm{d}X \sqrt{G}V(\phi_D+\psi) + \int \mathrm{d}X \sqrt{G}V(\psi) + \int \mathrm{d}X \sqrt{G}V'(\psi)\phi_D \right.$$
$$\left. + \frac{1}{2}\int \mathrm{d}X \sqrt{G}\phi_D V''(\psi)\phi_D\right) F(\phi_D+\psi). \tag{30}$$

For the proof let us shift variables in equation (18) and apply equations (9)–(12). Then,

$$\begin{aligned}
&\mathrm{d}\mu_D(\phi_D+\chi)\exp\left(-\int \mathrm{d}X\,\sqrt{G}V(\phi_D+\phi_B+\chi)\right)F(\phi_D+\phi_B+\chi)\\
&\quad=\mathrm{d}\mu_D(\phi_D))\exp\left(-\frac{1}{2}\int \mathrm{d}X\,\sqrt{G}\chi\mathcal{A}\chi\right)F(\phi_D+\psi)\\
&\quad\times\exp\left(-\int \mathrm{d}X\,\sqrt{G}\chi\mathcal{A}\phi_D-\int \mathrm{d}X\,\sqrt{G}V(\phi_D+\psi)\right)
\end{aligned}\tag{31}$$

where in the second step we inserted $\chi=\psi-\phi_B$ ($\chi=0$ on the boundary, hence the shift is admissible). Next, we make use of $\mathcal{A}\phi_B=0$ (then $\mathcal{A}\psi=\mathcal{A}\chi$), subtract the first two terms of the Taylor expansion of $V(\phi_D+\psi)$ in ψ and apply the formula for a Gaussian integral of an exponential of a quadratic form [16]:

$$\mathrm{d}\mu_D(\phi_D))\exp\left(-\frac{1}{2}\int \mathrm{d}X\,\sqrt{G}\phi_D V''(\psi)\phi_D\right)=(\det\mathcal{A}_\psi)^{-\frac{1}{2}}\,\mathrm{d}\mu_\psi(\phi_D).\tag{32}$$

The final result is expressed in equation (30). In this equation $\exp(-W(\psi))$ is the effective action in the tree approximation (discussed by [11]) and $\det\mathcal{A}_\psi^{-\frac{1}{2}}$ gives the one-loop approximation to the effective action in QFT with the boundary value Φ. The remaining $\mathrm{d}\mu_\psi(\phi_D)$ integral in equation (30) can be calculated in perturbation expansion. It starts with higher powers n ($n\geqslant 3$) of ϕ_D leading to corrections in higher loops to the effective action.

5. Conclusions

In this section, we derive some relations between correlation functions with respect to various measures discussed in earlier sections. Let us define

$$Z[\Phi]=\exp(-W_0(\phi_B))\int \mathrm{d}\mu_D(\phi_D)\exp(-W_I(\phi_D+\phi_B))\tag{33}$$

where ϕ_B is defined in equation (20) with Φ as a fixed boundary value. The definition (33) is introduced in such a way that it agrees with the large N formula of [2] and the semiclassical calculations of [11] and those in equation (30) (see also a discussion in [27]).

If $Z[\Phi]$ is the generating functional then there exists a field $\mathcal{O}(\mathbf{x})$ such that

$$Z[\Phi]=\left\langle\exp\left(\int_{\mathcal{B}}\mathcal{O}(x)\Phi(x)\sqrt{g}\,\mathrm{d}x\right)\right\rangle.\tag{34}$$

Treating $Z[\Phi]$ as the generating functional we can calculate

$$\begin{aligned}
&\frac{\delta}{\delta\Phi(\mathbf{x}_1)}\cdots\frac{\delta}{\delta\Phi(\mathbf{x}_n)}Z[\Phi]_{|\Phi=0}=\left(\mathcal{D}\frac{\delta}{\delta\phi_B}\right)(\mathbf{x}_1)\cdots\left(\mathcal{D}\frac{\delta}{\delta\phi_B}\right)(\mathbf{x}_n)\\
&\exp(-W_0(\phi_B))\int \mathrm{d}\mu_D(\phi_D)\exp(-W_I(\phi_D+\phi_B))|_{\phi_B=0}
\end{aligned}\tag{35}$$

where

$$\left(\mathcal{D}\frac{\delta}{\delta\phi_B}\right)(\mathbf{x})\equiv\int \mathrm{d}X\,\sqrt{G}\mathcal{D}(X,\mathbf{x})\frac{\delta}{\delta\phi_B(X)}$$

and

$$W_0(\phi_B)=\tfrac{1}{2}(\phi_B,\mathcal{A}\phi_B).$$

We wish to compare these correlation functions with those of the bulk field ϕ defined by the generating functional

$$S[J] = \int d\mu_0(\phi) \exp(-W_I(\phi) + (J, \phi)). \tag{36}$$

Then, the correlation functions can be calculated from the formula

$$\begin{aligned} &\frac{\delta}{\delta J(X_1)} \cdots \frac{\delta}{\delta J(X_n)} Z[J]_{|J=0} = \left(\mathcal{G}\frac{\delta}{\delta\phi_c}\right)(X_1)\cdots\left(\mathcal{G}\frac{\delta}{\delta\phi_c}\right)(X_n) \\ &\exp\left(\frac{1}{2}(\phi_c, \mathcal{A}\phi_c)\right)\int d\mu_0(\phi)\exp(-W_I(\phi+\phi_c))_{|\phi_c=0} \end{aligned} \tag{37}$$

where on the rhs we have absorbed the linear term of the exponential (36) into a shift of the measure according to equations (9) and (10) with $\phi_c = \mathcal{G}J$ and $J = \mathcal{A}\phi_c$. It is clear from equations (35) and (37) that a perturbative calculation of n-point correlation functions of $\mathcal{O}$ and ϕ involves the same graphs and only the propagators are different. The relation between (35) and (37) has been discovered earlier by Duetsch and Rehren [22] (see also [28]). A Hamiltonian derivation of the relation between differentiation with respect to boundary values and sources J can be found in [23].

The field theory in the bulk (6)–(7) is an integral over $Z[\Phi]$

$$Z_0 = \int d\nu_B(\Phi)\exp(W_0(\phi_B))Z[\Phi]. \tag{38}$$

We can obtain a connection between some other correlation functions. Generalizing equation (38) let us define the generating functional $S_D[\phi_B; J]$ in the ϕ_D theory shifted by a background field ϕ_B

$$S_D[\phi_B; J] = \int d\mu_D(\phi_D)\exp(-W_I(\phi_D+\phi_B))\exp\left(\int dX\sqrt{G}J\phi_D\right). \tag{39}$$

Then, from equation (18) the generating functional for correlation functions of the fields ϕ in the model (6) is

$$S[J] = \int d\mu_B(\phi_B)\exp\left(\int dX\sqrt{G}J\phi_B\right)S_D[\phi_B; J]. \tag{40}$$

It can be seen that ϕ_D and ϕ_B enter symmetrically in equation (18). Hence, we may also write

$$S[J] = \int d\mu_D(\phi_D)\exp\left(\int dX\sqrt{G}J\phi_D\right)S_B[\phi_D; J]. \tag{41}$$

Differentiating both sides of equations (40) and (41) we obtain a relation between correlation functions of the fields ϕ, ϕ_D and ϕ_B. The form of the correlation functions in the model (6) at the boundary points $\mathbf{x}_j \in \mathcal{B}$ is a simple consequence of equation (41)

$$\langle\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle = \int d\mu_D(\phi_D)\int d\nu_B(\Phi)\exp(-W_I(\phi_D+\phi_B))\Phi(\mathbf{x}_1)\cdots\Phi(\mathbf{x}_n). \tag{42}$$

In particular, if the interaction is concentrated only on the boundary

$$W_I(\phi) = \int_{\mathcal{B}} d\mathbf{x}\sqrt{g}V(\phi(0,\mathbf{x}))$$

then $\phi_D = 0$ in W_I in equation (42) and the functional integral (42) is the same as in QFT on $\mathcal{B}$ defined by the free field measure $d\nu_B$ with the covariance $\mathcal{G}_E(\mathbf{x}, \mathbf{x}')$. In the case of the hyperbolic space this covariance is logarithmic. Hence, ultraviolet problem is the same as for quantum fields in two dimensions.

We think that the QFT theory on a boundary of a curved manifold is interesting for itself because of its remarkable regularity expressed (for the hyperbolic space) in the strong decay (25) in the momentum space. However, the main result of this paper is formulated in equations (40)–(42). The formulae connecting the correlation functions of fields in various field theoretic models can shed some light on relations of the AdS-CFT type.

Acknowledgments

The author is grateful to an anonymous referee for pointing out [22]. The formulae at the beginning of section 5 resulted from my effort to derive the results of Duetsch and Rehren in the continuum as the addendum to an earlier version of my paper.

References

[1] Maldacena J 2002 *Lectures on large N field theories and gravity* Les Houches 76 (Berlin: Springer)
[2] Gubser S S, Klebanov I R and Polyakov A M 1998 *Phys. Lett.* B **428** 105 (*Preprint* hep-th/9802109)
[3] Witten E 1998 *Adv. Theor. Math. Phys.* **2** 25 (*Preprint* hep-th/9802150)
[4] Ford L H and Parker L 1977 *Phys. Rev.* D **16** 245
[5] Fronsdal C 1974 *Phys. Rev.* D **10** 589
[6] Bunch T S and Davies P C W 1978 *J. Phys. A: Math. Gen.* **11** 1315
[7] Schomblond Ch and Spindel P 1976 *Ann. Inst. Henri Poincare* **25** 67
[8] Birrell N D and Davies P C W 1982 *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press)
[9] Guth A and Pi S-Y 1985 *Phys. Rev.* D **32** 1899
[10] Tsamis N C and Woodard R P 1994 *Commun. Math. Phys.* **162** 217
[11] Mück W and Viswanathan K S 1998 *Phys. Rev.* D **58** 041901 (*Preprint* hep-th/9804035)
[12] Kato T 1976 *Perturbation Theory For Linear Operators* (Berlin: Springer)
[13] Guerra F, Rosen L and Simon B 1976 *Ann. Inst. Henri Poincare* **25** 231
[14] Simon B 1974 *The* $P(\phi)_2$ *Euclidean(Quantum) Field Theory* (Princeton, NJ: Princeton University Press)
[15] Cramer H and Leadbetter M R 1967 *Stationary and Related Stochastic Processes* (New York: Willey)
[16] Skorohod A V 1974 *Integration in Hilbert Space* (Berlin: Springer)
[17] Arefeva I Ya and Volovich I V 1998 *Preprint* hep-th/9809188
[18] Glimm J and Jaffe A 1981 *Quantum Physics* (Berlin: Springer)
[19] Gilbarg D and Trudinger N S 1983 *Elliptic Partial Differential Equations of Second Order* (Berlin: Springer)
[20] Haba Z 2004 *Mod. Phys. Lett.* A **19** 2411
[21] Haba Z 2005 *J. Math. Phys.* **46** 042301
[22] Duetsch M and Rehren K-H 2002 *Lett. Math. Phys.* **62** 171 (*Preprint* hep-th/0204123)
[23] Marolf D 2005 *J. High Energy Phys.* JHEP05(2005)042 (*Preprint* hep-th/0412032)
[24] Hawking S 1982 *Phys. Lett.* B **115** 295
[25] Dolgov A D, Einhorn M B and Zakharov V I 1995 *Phys. Rev.* D **52** 717
[26] Allen B 1985 *Phys. Rev.* D **32** 3136
Allen B and Folacci A 1987 *Phys. Rev.* D **35** 3771
[27] Rehren K-H 2004 QFT lectures on AdS-CFT *Preprint* hep-th/0411086
[28] Banks T, Douglas M R, Horowitz G T and Martinec E J 1998 AdS dynamics from conformal field theory *Preprint* hep-th/9808016
[29] Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
[30] Berenstein D, Corrado R, Fischler W and Maldacena J 1998 *Phys. Rev.* D **59** 105023 (*Preprint* hep-th/9809188)